

Multilinear singular integrals

by

JONATHAN COHEN (Athens, Ga.)

Abstract. This paper uses Fourier Transform and Mellin Transform analysis to obtain L^p estimates for certain multilinear singular integrals. The results obtained here extend estimates by Calderón, Coifman and Meyer on commutators of singular integrals to a wider class of multilinear singular integrals.

§ 1. Introduction. In this paper sharp estimates are obtained for operators of the type:

$$(1.1) \quad p.v. \int \prod_{j=1}^n \left\{ \frac{r_{m_j}(A_j; x, y)}{(x-y)^{m_j}} \right\} \frac{f(y)}{x-y} dy$$

where

$$r_{m_j}(A_j; x, y) = A_j(x) - \sum_{k=1}^{m_j-1} \frac{A_j^{(k)}(y)(x-y)^k}{k!}.$$

These operators are related to those introduced by Calderón in [2] and [3] and studied by Coifman and Meyer in [5], [6] and [7]. We will sometimes denote these operators by $D^N H \left\{ \prod_{j=1}^n r_{m_j}(A_j; x, \cdot) f(\cdot) \right\}$ where $D^N H$ is the Hilbert transform followed by the n th derivative. (The reason for this notation will become apparent in §3.)

The operators studied in this paper arise naturally from the study of higher commutators of differential and pseudo-differential operators. The simplest case is the commutator $[A, D^N H]$ where A is pointwise multiplication by the function $A(x)$. It has been shown by Calderón [3] that this commutator can be written as the sum of pseudo-differential operators of degree less than or equal to $N-1$ plus an operator of the type studied in this paper.

One can show that higher commutators of the form $[A_1, \dots, [A_n, D^N H] \dots]$ can be written as the sum of pseudo-differential operators of degree less than or equal to $N-n$ plus the sum of operators of the type considered in this paper. One example is the second commutator $[A, [B, D^N H]]$

which can be written

$$\begin{aligned}
 [A, [B, D^N H]]f &= D^N H[r_m(A; x, \cdot)r_n(B; x, \cdot)f(\cdot)] \\
 &+ \sum_{k=1}^{n-1} (-1)^k \binom{N}{k} D^{N-k} H[r_{N-k}(A; x, \cdot)B^{(k)}(\cdot)f(\cdot)] \\
 &+ \sum_{j=1}^{m-1} (-1)^j \binom{N}{j} D^{N-j} H[r_{N-j}(B; x, \cdot)A^{(j)}(\cdot)f(\cdot)] \\
 &+ \sum_{i=2}^{N-1} \sum_{j+k=i} (-1)^{j+k} \binom{N}{j,k} D^{N-j-k} H[A^{(j)}B^{(k)}f]
 \end{aligned}$$

where $m+n=N$, $j \geq 1$, $k \geq 1$ and $\binom{N}{j,k} = \frac{N!}{j!k!(N-j-k)!}$.

The results and methods used in this paper closely follow those developed by Coifman and Meyer in [6] and [7]. It appears as a natural follow up to [7] and the notation and organization have been chosen to be as consistent as possible with it.

Let $a_j \in \mathcal{S}$, $j=1, \dots, n$, $f \in \mathcal{S}$ (\mathcal{S} is the space of functions which are C^∞ and rapidly decreasing). Let $\sigma: R^{n+1} \rightarrow C$ be bounded and measurable. Assume $a = (a_1, \dots, a_n) \in R^n$, $s \in R$ and $(a, s) = (a_1, \dots, a_n, s) \in R^{n+1}$. We adopt the notation $\sigma(a, s) = \sigma(a_1, \dots, a_n, s)$, $\hat{a}(a) = \prod_{j=1}^n \hat{a}_j(a_j)$ and $da = \prod_{j=1}^n da_j$.

Let $P_n = \{(p_1, \dots, p_n, p_0): 1 < p_j < \infty; \sum_{j=1}^n (1/p_j) = 1/q < 1\}$. For $(p, p_0) = (p_1, \dots, p_n, p_0)$, we define

$$L^{(p, p_0)} = \{(a, f): a_j \in L^{p_j}, j=1, 2, \dots, n, f \in L^{p_0}\},$$

and

$$\|(a, f)\|_{(p, p_0)} = \left\{ \prod_{j=1}^n \|a_j\|_{p_j} \right\} \|f\|_{p_0}.$$

Using the above notation and assumptions on the functions a_j, f and σ we make the following definitions.

(1.3) DEFINITION.

$$(1) T_\sigma(a, f)(x) = \int_{R^{n+1}} e^{isx} \sigma(a, s) \hat{a}(a) \hat{f}\left(s - \sum_{j=1}^n a_j\right) da ds.$$

$$(2) \text{ For } (p, p_0) \in P_n, \sum_{j=0}^n (1/p_j) = \frac{1}{q} < 1,$$

$$M_n(p, p_0) = \{\sigma: \|T_\sigma(a, f)\|_q \leq c \|(a, f)\|_{(p, p_0)}\}$$

and

$$M_n = \bigcap_{(p, p_0) \in P_n} M_n(p, p_0).$$

The function σ will be called the *symbol of the operator* T_σ .

For $g(x)$ a function ($x \in \mathbb{R}^1$) with $m \geq 1$ derivatives, we define the Taylor series remainder operator

$$R_{-\alpha}^m g(x) = g(s - \alpha) - \sum_{k=0}^{m-1} \frac{g^{(k)}(s)(-\alpha)^k}{k!}.$$

We let $R_{-\alpha}^0 g(s) = g(s - \alpha)$.

If $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$, $0 \leq m_j$, $\alpha \in \mathbb{R}^n$ and g has $|m| = \sum_{j=1}^n m_j$ derivatives, we denote the n -fold composition of Taylor series of g by

$$R_{(-\alpha)}^{(m)} g(s) = R_{-\alpha_1}^{m_1} \dots R_{-\alpha_n}^{m_n} g(s).$$

Let

$$\omega_m(\alpha, s) = \frac{R_{(-\alpha)}^{(m)} s^{|m|} \operatorname{sgn} s}{\prod_{j=1}^n \alpha_j^{m_j}}.$$

The operator T_{ω_m} will be called a *commutator of order* $|m|$. We note that if $m_j = 1$ for $j = 1, 2, \dots, n$, then the operator T_{ω_m} is the n th commutator treated in the papers of Coifman and Meyer [6] and [7]. (The order of the operator T_ω does not have the same meaning as the order of a pseudo-differential operator.)

The main results of this paper are the following theorems.

THEOREM I. *If $a_1, \dots, a_n \in \mathcal{S}$, then*

$$\|T_{\omega_m}(a, f)\|_q \leq c \|f\|_{p_0} \prod_{j=1}^n \|a_j\|_{p_j}$$

where $1 > 1/q = \sum_{j=0}^n (1/p_j)$, $1 < p_j < \infty$, $j = 0, 1, \dots, n$.

THEOREM II. *For $a_j \in \mathcal{D}$ (C^∞ functions with compact support), $a_j =$*

$$A_j^{(m_j)} = \left(\frac{d}{dx}\right)^{m_j} A_j, \text{ and}$$

$$r_{m_j}(A_j; x, y) = A_j(x) - \sum_{k=0}^{m_j-1} \frac{A_j^{(k)}(y)(x-y)^k}{k!},$$

$$(1.4) \quad p.v. \int \prod_{j=1}^n \left\{ \frac{r_{m_j}(A_j; x, y)}{(x-y)^{m_j}} \right\} \frac{f(y)}{(x-y)} dy \\ = c_m \int e^{isx} \omega_m(\alpha, s) \hat{a}(\alpha) \hat{f}\left(s - \sum_{j=1}^n \alpha_j\right) da ds$$

$$\text{where } c_m = \frac{(-i\pi)(-1)^{|m|}}{|m|!(2\pi)^{n+1}}.$$

THEOREM III. We define $T_\varepsilon^m(a, f) = T_\varepsilon^{m_1, \dots, m_n}(a_1, \dots, a_n)$ by

$$T_\varepsilon^m(a, f)(x) = \int_{|x-y|>\varepsilon} \prod_{j=1}^n \left\{ \frac{r_{m_j}(A_j; x, y)}{(x-y)^{m_j}} \right\} \frac{f(y)}{x-y} dy$$

and let

$$T_*^m(a, f)(x) = \sup_{\varepsilon>0} |T_\varepsilon^m(a, f)(x)|.$$

Then, for $0 < q < \infty$

$$\int |T_*^m(a, f)(x)|^q dx \leq c \int \left\{ \left[\prod_{j=1}^n a_j^*(x) \right] f^*(x) \right\}^q dx$$

where a_j^* is the Hardy-Littlewood maximal function of a .

§2. An L^p estimate for smooth functions. We defined the order of the symbol ω_m to be $|m| = \sum_{j=1}^n m_j$ where $m = (m_1, \dots, m_n) \in Z^n$ and $0 \leq m_j$. The proof of Theorem 1 is by induction on the order of ω_m .

To clarify the inductive hypothesis we make the following observations.

- (1) Let $\sigma(a_1, \dots, a_n, s) \in M_n$. Then if $\tau(a_1, \dots, a_n, \beta, s) = \sigma(a_1, \dots, a_n, s)$, $\tau \in M_{n+1}$.
- (2) If $\sigma \in M_n$, then $R_{-\beta}^0 \sigma(a_1, \dots, a_n, s) = \sigma(a_1, \dots, a_n, s - \beta) \in M_{n+1}$.
- (3) The zero order symbols are those ω 's of the form $R_{-a_1}^0 \dots R_{-a_n}^0 \text{sgn } s = \text{sgn}(s - a_1 - \dots - a_n)$.

Applying observations (1)-(3) and the fact that $H\hat{f}(s) = (-i\pi)\text{sgn } s \hat{f}(s)$ where $Hf(x) = p.v. \int \frac{f(y)}{x-y} dy$, we see that the zero commutators are of the form $[\prod_{j=1}^{n-k} a_j][H(f \prod_{j=k+1}^n a_j)] \in M_n$.

The inductive hypothesis is the following: for $\bar{m} = (\bar{m}_1, \dots, \bar{m}_n) \in Z^n$, $0 \leq \bar{m}_j$ and $|\bar{m}| \leq |m| - 1$, the symbol $\omega_{\bar{m}} \in M_n$. From observations (1)-(3) it is clear that we need only show that the inductive hypothesis implies $\omega_m \in M_n$ when $m = (m_1, \dots, m_n) \in Z^n$, $\sum_{j=1}^n m_j = |m|$ and $m_j \geq 1$.

The symbol

$$\bar{\omega}_m = \frac{sR_{-a_1}^{m_1} \dots R_{-a_n}^{m_n} s^{|m|-1} \text{sgn } s}{\prod_{j=1}^n a_j^{m_j}}$$

is introduced and it is shown that $\omega_m - \bar{\omega}_m$ is a sum of lower order commutators. It is enough to show that $\bar{\omega}_m \in M_n$.

(2.1) DEFINITION. $\mathcal{S}^x = \{\varphi: \varphi(e^x) \in \mathcal{S}\}$.

(2.2) LEMMA. For all $\varphi \in \mathcal{S}^x$, $\sigma \in M_n$, and all integers j , $1 \leq j \leq n$, $\varphi(|s/\alpha_j|)\sigma(a, s) \in M_n$.

Proof. For $\varphi \in \mathcal{S}^x$,

$$\varphi(x) = \int_{-\infty}^{\infty} \hat{\psi}(\gamma)x^{i\gamma} d\gamma$$

where $\psi(x) = \varphi(e^x) \in \mathcal{S}$. Hence

$$\varphi(|s/\alpha_j|)\sigma(a, s) = \int_{-\infty}^{\infty} |s|^{i\gamma} |\alpha_j|^{-i\gamma} \sigma(a, s) \hat{\psi}(\gamma) d\gamma.$$

But $|s|^{i\gamma}$ is a bounded multiplier with norm not exceeding $(1+|\gamma|)$ so that the operator $T_{\varphi(|s/\alpha_j|)\sigma}$ has norm less than or equal to $\int_{-\infty}^{\infty} (1+|\gamma|)^2 \|\sigma\| \hat{\psi}(\gamma) d\gamma$ where $\|\sigma\|$ is the operator norm of T_σ .

Since \mathcal{S} is closed under the Fourier Transform, $\hat{\psi} \in \mathcal{S}$ and we have $\int_{-\infty}^{\infty} |\hat{\psi}(\gamma)|(1+|\gamma|)^N d\gamma < \infty$ for any positive integer N .

(2.3) LEMMA. Let $\tau: R^{n+1} \rightarrow C$ be a measurable function. Let \mathcal{C} be the set of $\gamma = (t_1, \dots, t_n)$ such that $|\alpha_1|^{t_1} \dots |\alpha_n|^{t_n} \tau(a, s) \in M_n$. Then \mathcal{C} is a convex set.

Proof. The proof is given in proposition 3 of [6]. Its importance to this paper is the following corollary.

(2.4) COROLLARY. Assume that $\omega_{m'} \in M_n$ for $|m'| < |m|$. Then for $0 \leq \theta_j \leq 1$ and $\sum_{j=1}^n \theta_j = 1$, we have,

$$|\alpha_1|^{\theta_1} \dots |\alpha_n|^{\theta_n} \frac{R_{-\alpha_1}^{m_1} \dots R_{-\alpha_n}^{m_n} s^{|m|-1} \operatorname{sgn} s}{\prod_{j=1}^n \alpha_j^{m_j}} \in M_n.$$

Proof. The proof only involves the case $\theta_1 = 1, \theta_j = 0$ for $j = 2, \dots, n$. The cases $\theta_j = 1, \theta_k = 0$ for $k \neq j$ are taken care of by a symmetric argument and the rest of the cases follow by convexity from Lemma (2.3). So one must show

$$(2.5) \quad |\alpha_j| \frac{R_{-\alpha_1}^{m_1} \dots R_{-\alpha_n}^{m_n} s^{|m|-1} \operatorname{sgn} s}{\prod_{j=1}^n \alpha_j^{m_j}} \in M_n.$$

A simple argument shows that if $\sigma(a, s) \in M_n$, then $\operatorname{sgn} a_j \sigma(a, s) \in M_n$. Hence we may replace $|\alpha_j|$ by a_j in (2.5).

If $m_1 = 1$, let $m' = (m_2, m_3, \dots, m_n)$. Then

$$\begin{aligned} & \frac{R_{-a_1}^{m_1} \dots R_{-a_n}^{m_n} s^{|m|-1} \operatorname{sgn} s}{\prod_{j=1}^n a_j^{m_j}} \\ &= \frac{R_{-a_2}^{m_2} \dots R_{-a_n}^{m_n} (s - a_1)^{|m|-1} \operatorname{sgn}(s - a_1)}{\prod_{j=2}^n a_j^{m_j}} - \frac{R_{-a_2}^{m_2} \dots R_{-a_n}^{m_n} s^{|m|-1} \operatorname{sgn} s}{\prod_{j=2}^n a_j^{m_j}}. \end{aligned}$$

Calling the first term σ_1 and the second term σ_2 we see that $T_{\sigma_1}(a, f) = a_1 T_{\omega_m}(a, f)$ and $T_{\sigma_2}(a, f) = T_{\omega_m}(a, a_1 f)$. Both T_{σ_1} and T_{σ_2} satisfy Definition (1.3) and so they are both in M_n .

If $m_1 > 1$, let $m' = (m_2, \dots, m_n)$ as above and let $m'' = (m_1 - 1, m_2, \dots, m_n)$. Then once again,

$$\begin{aligned} & \frac{R_{-a_1}^{m_1} \dots R_{-a_n}^{m_n} s^{|m|-1} \operatorname{sgn} s}{\prod_{j=1}^n a_j^{m_j}} = \frac{R_{-a_1}^{m_1} R_{-a_2}^{m_2} \dots R_{-a_n}^{m_n} s^{|m|-1} \operatorname{sgn} s}{a_1^{m_1-1} \prod_{j=2}^n a_j^{m_j}} \\ & - (-1)^{m_1-1} \frac{(|m|-1)!}{(m_1-1)! |m'|!} \frac{R_{-a_2}^{m_2} \dots R_{-a_n}^{m_n} s^{|m'|} \operatorname{sgn} s}{\prod_{j=2}^n a_j^{m_j}}. \end{aligned}$$

The first term is the symbol of the operator $T_{\omega_m}(a, f)$ and the second is a constant times the symbol of $T_{\omega_{m'}}(a, a_1 f)$. Both of these are commutators of order lower than $|m|$ and so satisfy Definition (1.3) by induction. This proves the corollary.

(2.6) LEMMA. $\bar{\omega}_m - \omega_m \in M_n$ where

$$\bar{\omega}_m = \frac{s R_{-a_1}^{m_1} \dots R_{-a_n}^{m_n} s^{|m|-1} \operatorname{sgn} s}{\prod_{j=1}^n a_j^{m_j}}.$$

Proof. Lemma (2.6) follows from the following identity. For $\varphi(s)$ a function with $|m|$ derivatives,

$$\begin{aligned} (2.7) \quad & R_{-a_1}^{m_1} \dots R_{-a_n}^{m_n} s \varphi(s) \\ &= s R_{-a_1}^{m_1} \dots R_{-a_n}^{m_n} \varphi(s) - \sum_{j=1}^n a_j R_{-a_1}^{m_1} \dots R_{-a_j}^{m_j-1} \dots R_{-a_n}^{m_n} \varphi(s). \end{aligned}$$

Hence,

$$\bar{\omega}_m - \omega_m = \sum_{j=1}^n \frac{R_{-a_1}^{m_1} \dots R_{-a_j}^{m_j-1} \dots R_{-a_n}^{m_n} s^{|m|-1} \operatorname{sgn} s}{a_j^{m_j-1} \prod_{\substack{k \neq j \\ k=1}}^n a_k^{m_k}}$$

and all of the terms in this sum are in M_n since they are the symbols of commutators of order less than $|m|$.

(2.8) LEMMA. Let $\eta \in C_0^\infty$ where η is even, non-negative, at most one and satisfies $\eta(t) \equiv 1$ for $t \in [-n, n]$ and $\eta(t) \equiv 0$ for $t \notin [-2n, 2n]$. Then $\bar{\omega}_m(\alpha, s) \prod_{j=1}^n (1 - \eta(s/\alpha_j)) \equiv 0$.

Proof. If $1 - \eta(s/\alpha_j) \neq 0$ for all j , then $|s| > n|\alpha_j|$ for all j and so $\text{sgn}(s - \sum_{j \in J} \alpha_j) = \text{sgn } s$ for any subset J of $\{1, 2, \dots, n\}$. Thus $\prod_{j=1}^n (1 - \eta(s/\alpha_j)) \neq 0$ implies that

$$\bar{\omega}_m(\alpha, s) = s \text{sgn } s \frac{R_{-\alpha_1}^{m_1} \dots R_{-\alpha_n}^{m_n} s^{|m|-1}}{\prod_{j=1}^n \alpha_j^{m_j}} \equiv 0.$$

Proof of Theorem I. To prove Theorem I it suffices to show that $\bar{\omega}_m \in M_n$ since by Lemma (2.6) $\bar{\omega}_m - \omega_m \in M_n$. From Lemma (2.8),

$$\bar{\omega}_m(\alpha, s) = - \sum_{J \in \mathcal{J}_0} (-1)^{|J|} \prod_{j \in J} \eta(s/\alpha_j) \bar{\omega}_m(\alpha, s)$$

where \mathcal{J}_0 is the set of non-empty subsets of $\{1, 2, \dots, n\}$ and $|J|$ is the number of elements in J . If we define

$$\omega_J(\alpha, s) = \prod_{j \in J} \eta(s/\alpha_j) \bar{\omega}_m(\alpha, s),$$

then

$$\omega_J = \text{sgn } s \left\{ \prod_{j \in J} \eta(|s/\alpha_j|) |s/\alpha_j|^{|J|} \right\} \left\{ \left[\prod_{j \in J} |\alpha_j|^{|J|} \right] \frac{R_{-\alpha_1}^{m_1} \dots R_{-\alpha_n}^{m_n} s^{|m|-1} \text{sgn } s}{\prod_{k=1}^n \alpha_k^{m_k+1}} \right\}.$$

Let

$$\tilde{\omega}_J = \left\{ \left[\prod_{j \in J} |\alpha_j|^{|J|} \right] \frac{R_{-\alpha_1}^{m_1} \dots R_{-\alpha_n}^{m_n} s^{|m|-1} s \text{sgn } s}{\prod_{k=1}^n \alpha_k^{m_k+1}} \right\}.$$

Then $\tilde{\omega}_J \in M_n$ by Corollary (2.4). $\tilde{\omega}_J \prod_{j \in J} \eta(|s/\alpha_j|) |s/\alpha_j|^{|J|} \in M_n$ by Lemma (2.2) and the fact that $t^{|J|} \eta(t) \in \mathcal{S}^x$. Finally, summing up the J 's we get that $\bar{\omega}_m \in M_n$.

§ 3. The fundamental identity. The purpose of this section is to show that the operators considered in §2 can be realized as singular integrals if enough regularity is assumed on the α_j 's and f . Specifically, for $\alpha_j \in \mathcal{D}, j = 1, 2, \dots, n, f \in \mathcal{D}, m = (m_1, \dots, m_n)$ and $N = \sum_{j=1}^n m_j, T_{\omega_m}$

$= D^N H \left[\prod_{j=1}^n r_{m_j}(A_j; x, \cdot) f(\cdot) \right]$ and this operator is given by

$$\frac{1}{c_m} p.v. \int \prod_{j=1}^n \left\{ \frac{r_{m_j}(A_j; x, y)}{(x-y)^{m_j}} \right\} \frac{f(y)}{x-y} dy \quad \text{where} \quad c_m = \frac{(-i\pi)(-1)^{|m|}}{|m|!(2\pi)^{n+1}}.$$

(3.1) LEMMA. Assume $g \in \mathcal{D}$ and $x \in \mathbb{R}$.

(1) If N is a positive integer and $g(x) = g^{(1)}(x) = \dots = g^{(N-1)}(x) = 0$, then

$$(3.2) \quad \left(\frac{d}{dx} \right)^N \left\{ p.v. \int \frac{g(y)}{x-y} dy \right\} = (-1)^N N! p.v. \int \frac{g(y)}{(x-y)^{N+1}} dy.$$

(2) If k, N are positive integers with $k \leq N$, then

$$(3.3) \quad \left(\frac{d}{dx} \right)^N \left\{ p.v. \int \frac{g(y)(x-y)^k}{(x-y)} dy \right\} = \frac{(-1)^k N!}{(N-k)!} \left(\frac{d}{dx} \right)^{N-k} \left\{ p.v. \int \frac{g(y)}{x-y} dy \right\}.$$

Proof. Choose $M > 0$ large enough so that the interval $[-M, M]$ contains the support of g . Let $\eta \in \mathcal{D}$ be an even function where $0 \leq \eta(t) \leq 1$, $\eta(t) \equiv 1$ for $t \in [-4M, 4M]$ and $\eta(t) \equiv 0$ for $t \notin [-8M, 8M]$. Let $\eta_k(t) = \eta(t/k)$. For $k \geq 1$ we clearly have for $x \in [-2M, 2M]$

$$p.v. \int \frac{g(y)}{x-y} dy = \int \frac{g(y) - g(x)}{x-y} \eta_k(x-y) dy.$$

By differentiating N times and letting $k \rightarrow \infty$ we get the formula

$$\left(\frac{d}{dx} \right)^N \left\{ p.v. \int \frac{g(y)}{(x-y)} dy \right\} = (-1)^N N! \lim_{\delta \rightarrow \infty} \int_{|x-y| \leq \delta} \frac{g(y) - \sum_{j=1}^N \frac{g^{(j)}(x)(y-x)^j}{j!}}{(x-y)^{N+1}} dy.$$

If $g(x) = g^{(1)}(x) = \dots = g^{(N-1)}(x) = 0$, formula (3.4) yields

$$\lim_{\delta \rightarrow \infty} \int_{|x-y| \leq \delta} \frac{g(y) - \frac{g^{(N)}(x)(y-x)^N}{N!}}{(x-y)^{N+1}} dy = p.v. \int \frac{g(y)}{(x-y)^{N+1}} dy$$

which establishes (3.2).

Applying (3.4) to the function $\psi(y) = g(y)(x-y)^k$ (where k, N are positive integers with $k \leq N$) establishes (3.3).

(3.5) Remark. Assume $A_1, \dots, A_n \in \mathcal{S}$. Then

$$\left(\frac{d}{dy} \right)^N \prod_{j=1}^n r_{m_j}(A_j; x, y)|_{y=x} = 0$$

for $N = 0, 1, \dots, |m| - 1$ where $|m| = \sum_{j=1}^n m_j$.

Remark (3.5) and Lemma (3.1) show that

$$D^{|m|} H \left[\prod_{j=1}^n r_{m_j}(A_j; x, \cdot) f(\cdot) \right] = (-1)^{|m|} |m|! \text{p.v.} \int \prod_{j=1}^n \left\{ \frac{r_{m_j}(A_j; x, y)}{(x-y)^{m_j}} \right\} \frac{f(y)}{x-y} dy.$$

(3.6) LEMMA. Assume $f \in \mathcal{D}$, $A_1, \dots, A_n \in \mathcal{S}$. Then

$$(3.7) \quad i^{|m|} C_m \int_{\mathbb{R}^{n+1}} R_{(-a)}^{(m)} s^{|m|} \text{sgn } s \hat{A}(\alpha) \hat{f} \left(s - \sum_{j=1}^n \alpha_j \right) e^{isx} d\alpha ds = \text{p.v.} \int \prod_{j=1}^n \left\{ \frac{r_{m_j}(A_j; x, y)}{(x-y)^{m_j+1}} \right\} \frac{f(y)}{x-y} dy$$

where C_m is as in Theorem II.

Proof. To show this we introduce some multiple index notation:

\mathcal{J} = set of subsets of $\{1, 2, \dots, n\}$,

\mathcal{J}' = complement of \mathcal{J} in $\{1, 2, \dots, n\}$,

$|\mathcal{J}|$ = number of elements in \mathcal{J} ,

$k_{\mathcal{J}} = (k_{j_1}, \dots, k_{j_t})$ where $\mathcal{J} = \{j_1, \dots, j_t\}$ and $j_1 < j_2 < \dots < j_t$,

$K_{\mathcal{J}} = \{k_{\mathcal{J}} : 0 \leq k_{j_1} \leq m_{j_1} - 1; \dots; 0 \leq k_{j_t} \leq m_{j_t} - 1\}$,

$k_{\mathcal{J}}! = k_{j_1}! k_{j_2}! \dots k_{j_t}!$,

$|k_{\mathcal{J}}| = k_{j_1} + k_{j_2} + \dots + k_{j_t}$,

$g^{(k_{\mathcal{J}})}(x) = g^{(k_{j_1} + \dots + k_{j_t})}(x)$.

Using this notation we have the expansion for any $g \in \mathcal{S}$

$$(3.8) \quad R_{(-a)}^{(m)} g(s) = \sum_{\mathcal{J} \in \mathcal{S}} \sum_{k_{\mathcal{J}} \in K_{\mathcal{J}}} \frac{(-1)^{|\mathcal{J}|} g^{(k_{\mathcal{J}})} \left(s - \sum_{j \in \mathcal{J}'} \alpha_j \right)}{k_{\mathcal{J}}!} \prod_{j \in \mathcal{J}} (-\alpha_j)^{k_j}.$$

For $a \in \mathbb{R}^n$, $x \in \mathbb{R}$, we use the following notation:

$$\hat{A}_{\mathcal{J}}(a) = \prod_{j \in \mathcal{J}} \hat{A}_j(\alpha_j),$$

$$A_{\mathcal{J}}^{(k_{\mathcal{J}})}(x) = \prod_{j \in \mathcal{J}} A_j^{(k_j)}(x)$$

and

$$da = \prod_{j \in \mathcal{J}} d\alpha_j.$$

With this additional notation, setting $g(s) = s^{|m|} \operatorname{sgn} s$ and using (3.8)

$$\begin{aligned}
 (3.9) \quad & \int_{R^{n+1}} R_{(-a)}^{(m)} s^{|m|} \operatorname{sgn} s \hat{A}(a) \hat{f}\left(s - \sum_{j=1}^n a_j\right) e^{isx} da ds \\
 &= \sum_{J \in \mathcal{J}} \sum_{k_J \in K_J} \frac{(-1)^{|m| - |k_J| + |J|}}{(-i\pi)} \cdot \frac{i^{|m|} |m|!}{(|m| - |k_J|)!} \int_R e^{isx} \times \\
 & \times \int_{R^{|J|}} \left[i \left(s - \sum_{j \in J} a_j \right) \right]^{|m| - |k_J|} (-i\pi) \operatorname{sgn} \left(s - \sum_{j \in J} a_j \right) \hat{A}_{J'}(a) da_{J'} \times \\
 & \times \int_{R^{|J|}} \frac{A_J^{(k_J)} \hat{}(a)}{k_J!} \hat{f}\left(s - \sum_{j=1}^n a_j\right) da_J ds.
 \end{aligned}$$

Integrating in $a_{J'}$, the inner integral equals

$$(2\pi)^{|J|} \left\{ \frac{f A_J^{(k_J)} \hat{}}{k_J!} \right\} \left(s - \sum_{j \in J} a_j \right).$$

Integrating next in a_J , we get the middle integral is equal to

$$(2\pi)^n \left\{ A_{J'} D^{|m| - |k_J|} H \left(\frac{f A_J^{(k_J)} \hat{}}{k_J!} \right) \right\} (s).$$

Integrating in s and using Lemma (3.1) and Remark (3.5), the right-hand side of (3.9) becomes

$$\begin{aligned}
 & \sum_{J \in \mathcal{J}} \sum_{k_J \in K_J} \frac{(-i)^{|m|} (2\pi)^{n+1}}{-i\pi} (-1)^{|J|} \prod_{j \in J} A_j(x) D^{|m|} H \left\{ f(\cdot) \prod_{j \in J} \frac{A_j^{(k_j)}(\cdot)}{k_j!} (x - \cdot)^{k_j} \right\} (x) \\
 &= i^{|m|} C_m p.v. \int \frac{\prod_{j=1}^n r_{m_j}(A_j; x, y)}{(x-y)^{|m|+1}} f(y) dy.
 \end{aligned}$$

Proof of Theorem II. Let

$$I(\varepsilon) = i^{|m|} C_m \int_{R^{n+1}} e^{isx} \frac{R_{(-a)}^{(m)} s^{|m|} \operatorname{sgn} s}{\prod_{j=1}^n (i a_j + \varepsilon)^{m_j}} \hat{a}(a) \hat{f}\left(s - \sum_{j=1}^n a_j\right) da ds.$$

If $A_j^\varepsilon(x) = e^{-\varepsilon x} \int_{-\infty}^x \int_{-\infty}^{t_1} \dots \int_{-\infty}^{t_{m_j-1}} a_j(s) e^{\varepsilon s} ds$, then $A_j^\varepsilon \in \mathcal{S}$ and $A_j^\varepsilon(a_j) = \frac{\hat{a}_j(a_j)}{(i a_j + \varepsilon)^{m_j}}$. Applying Lemma (3.6)

$$I(\varepsilon) = p.v. \int \prod_{j=1}^n \left\{ \frac{r_{m_j}(A_j^\varepsilon; x, y)}{(x-y)^{m_j}} \right\} \frac{f(y)}{x-y} dy.$$

and

$$(a, f)^{(*)}(x) = \left\{ \prod_{j=1}^n a_j^*(x) \right\} f^*(x).$$

Next define the truncated operator

$$T_\epsilon^m(a, f)(x) = \int_{|x-y|>\epsilon} t_m(x, y) f(y) dy$$

where

$$t_m(x, y) = \frac{1}{x-y} \prod_{j=1}^n \frac{r_{m_j}(A_j; x, y)}{(x-y)^{m_j}} \quad \text{and} \quad T_*^m(a, f)(x) = \sup_{\epsilon>0} |T_\epsilon^m(a, f)(x)|.$$

We are now ready to state the principal result of this section.

(4.1) PROPOSITION. *If $a_j \in L^1$ for $j = 1, \dots, n$, $f \in L^1$, then there exists γ_0 so that for $\gamma < \gamma_0$,*

$$|x: T_*^m(a, f)(x) > 2\lambda, (a, f)^{(*)}(x) \leq \gamma\lambda| \leq c\gamma^{n+1} |x: T_*^m(a, f)(x) > \lambda|.$$

(The notation $|x: \dots|$ means the measure of the set x such that "...".)

Theorem 1 will follow as an immediate consequence of Proposition (4.1). The proof will be given in a series of lemmas, following closely the real variable methods used by Coifman and Meyer in [5].

(4.2) LEMMA. *If $|x-y| > \epsilon$, $|x-x_1| < \epsilon/4$, then for $x_2 \in (x-\epsilon/4, x+\epsilon/4)$ we have the following estimates.*

$$(a) \quad |t_m(x, y) - t_m(x_1, y)| < c \frac{a^{(*)}(x_2) |x-x_1|}{|x-y|^2},$$

$$(b) \quad \left| \int_{|x-y|>\epsilon} [t_m(x, y) - t_m(x_1, y)] f(y) dy \right| \leq c(a, f)^{(*)}(x_2)$$

and for $|y-y_1| < \frac{1}{2}|x-y|$, $|y-y_2| < \frac{1}{2}|x-y|$,

$$(c) \quad |t_m(x, y) - t_m(x, y_1)| \leq ca^{(*)}(y_2) \frac{|y-y_1|}{|x-y|^2}.$$

Proof.

$$\begin{aligned} t_m(x, y) - t_m(x_1, y) &= \prod_{j=1}^n r_{m_j}(A_j; x, y) \left[\frac{1}{(x-y)^{|m|+1}} - \frac{1}{(x_1-y)^{|m|+1}} \right] + \\ &+ \frac{1}{(x_1-y)^{|m|+1}} \sum_{j=1}^n [r_{m_j}(A_j; x, y) - r_{m_j}(A_j; x_1, y)] \prod_{k<j} r_{m_k}(A_k; x, y) \times \\ &\quad \times \prod_{k>j} r_{m_k}(A_k; x_1, y) \\ &\leq ca^{(*)}(x_2) \frac{|x-x_1|}{|x-y|^2} + c \sum_{j=1}^n a_j^*(x_2) \frac{|x-x_1|}{|x-y|^2} \prod_{k \neq j} a_k^*(x_2) \end{aligned}$$

$$\leq ca^{(*)}(x_2) \frac{|x-x_1|}{|x-y|^2}.$$

A standard argument tells us that for $|x-x_2| \leq |x-x_1|$,

$$\int_{|x-y| > 2|x-x_1|} \frac{|x-x_1|}{|x-y|^2} |f(y)| dy \leq cf^*(x_2)$$

which implies part (b) of Lemma (4.2). Finally, part (c) follows by the same argument as (a).

Before stating the next lemma it is necessary to introduce a Hardy-Littlewood maximal function and an appropriate multi-index notation.

For $(a, f) = (a_1, a_2, \dots, a_n, f)$ define

$$A_{p_j}(a_j)(x) = \sup_{I \ni x} \left\{ \frac{1}{|I|} \int_I |a_j(t)|^{p_j} dt \right\}^{1/p_j}$$

and the corresponding multi-index notation

$$A_{(p)}(a)(x) = \prod_{j=1}^n A_{p_j}(a_j)(x),$$

$$A_{(p, p_0)}(a, f)(x) = A_{(p)}(a) A_{p_0}(f)(x).$$

(4.3) LEMMA. Assume $(a, f) \in L^{(p, p_0)}$, $p_j \geq 1$, $j = 1, 2, \dots, n$ and $1 \leq p_0 < \infty$. Further, assume $0 < \delta < q$ where $1/q = \sum_{j=0}^n (1/p_j)$. If $T^m(a, f)$ satisfies the weak type inequality:

$$(4.4) \quad |x: T^m(a, f)(x) > \lambda| \leq c \left[\frac{\|(a, f)\|_{(p, p_0)}}{\lambda} \right]^q,$$

then the maximal function $T_*^m(a, f)$ satisfies

$$T_*^m(a, f)(x) \leq c [A_\delta(T^m(a, f))(x) + A_{(p, p_0)}(a, f)(x)]$$

and the same weak type inequality as in (4.4) is valid for $T_*^m(a, f)$.

Proof. Before proving Lemma (4.3) we note that Theorems I and II imply (4.4) if $\sum_{j=0}^n \left(\frac{1}{p_j}\right) < 1$, $1 < p_j < \infty$ and the a_j 's and f are C^∞ with compact support.

The lemma is proved for $a_j \in \mathcal{D}$, $j = 1, \dots, n$, $f \in \mathcal{D}$ and then it follows from standard arguments that the lemma can be extended to any $(a, f) \in L^{(p, p_0)}$.

For $\varepsilon > 0$, let χ_ε be the characteristic function of the interval $[x-\varepsilon, x+\varepsilon]$. Let $\eta_\varepsilon \in \mathcal{D}$ be a function which is one on the interval $[x-e, x+e]$,

less than or equal to one, non-negative and everywhere vanishing outside the interval $[x-2\varepsilon, x+2\varepsilon]$. Then, for $x_1 \in [x-\varepsilon/4, x+\varepsilon/4]$, we get the identity

$$(4.5) \quad T_\varepsilon^m(a, f)(x) = \int [t_m(x, y) - t_m(x_1, y)][1 - \eta_\varepsilon(y)]f(y) dy + \\ + \int t_m(x_1, y)f(y) dy - \int t_m(x_1, y)\eta_\varepsilon(y)f(y) dy + \\ + \int t_m(x, y)[\eta_\varepsilon(y) - \chi_\varepsilon(y)]f(y) dy.$$

Taking absolute values, raising to the δ power, averaging in x_1 over the interval $[x-\varepsilon/4, x+\varepsilon/4]$ and taking the $1/\delta$ power gives the estimate

$$(4.6) \quad |T^m(a, f)(x)| \\ \leq c \left\{ (a, f)^{(*)}(x) + A_\delta |T^m(a, f)(x)| + \sup_{\varepsilon > 0} \left(\frac{2}{\varepsilon} \int_{x-\varepsilon/4}^{x+\varepsilon/4} |T^m(a, \eta_\varepsilon f)(x_1)|^\delta dx_1 \right)^{1/\delta} \right\}.$$

The $(a, f)^{(*)}(x)$ comes from the first and last term of (4.5) after applying Lemma (4.2). That is,

$$\left| \int [t_m(x, y) - t_m(x_1, y)][1 - \eta_\varepsilon(y)]f(y) dy \right| \\ \leq \int |t_m(x, y) - t_m(x_1, y)|[1 - \chi_\varepsilon(y)]|f(y)| dy \leq c(a, f)^{(*)}(x)$$

and

$$\left| \int t_m(x_1, y)[\eta_\varepsilon(y) - \chi_\varepsilon(y)]f(y) dy \right| \\ \leq ca^{(*)}(x) \int_{2\varepsilon > |x-y| > \varepsilon} \frac{|x-x_1|}{|x-y|^2} |f(y)| dy \leq c(a, f)^{(*)}(x).$$

Both estimates are independent of x_1 and are left unaffected by averaging in x_1 .

To evaluate the second term on the right hand side of (4.6) we use the fact that the usual Hardy-Littlewood maximal function is weak type 1-1. If we set $E_\lambda = \{x: |T^m(a, f)(x)|^{\delta^*} > \lambda^{\delta^*}\}$, then

$$|E_\lambda| \leq \frac{2}{\lambda^\delta} \int_{E_\lambda} |T^m(a, f)(x)|^\delta dx$$

using Kolmogorov's inequality

$$\leq \frac{2}{\lambda^\delta} |E|^{1-\delta/\alpha} \|(a, f)\|_{(p, p_0)}^\delta.$$

Taking the $1/\delta$ power of both sides and simplifying we get

$$(4.7) \quad |E_\lambda| \leq \left\{ \frac{2 \|(a, f)\|_{(p, p_0)}^\delta}{\lambda} \right\}^\alpha.$$

To evaluate the last term in (4.6) we need to be a little careful. We let $\eta_{2\epsilon} = \eta_\epsilon(t/2)$. Then $\eta_{2\epsilon}(t) \equiv 1$ for $|t-x| < 2\epsilon$. If we adopt the notation $\eta_{2\epsilon}a = (\eta_{2\epsilon}a_1, \dots, \eta_{2\epsilon}a_n)$, then for $|x_1-x| < \epsilon/4$,

$$T^m(a, \eta_\epsilon f)(x_1) = T^m(\eta_{2\epsilon}a, \eta_\epsilon f)(x_1).$$

Applying Kolmogorov's inequality again (the weak type property is guaranteed since $\eta_{2\epsilon}a_j \in \mathcal{D}$ and $\eta_\epsilon f \in \mathcal{D}$) we have,

$$\begin{aligned} & \left\{ \frac{2}{\epsilon} \int_{x-\epsilon/4}^{x+\epsilon/4} |T^m(\eta_{2\epsilon}a, \eta_\epsilon f)(x_1)|^\delta dx_1 \right\}^{1/\delta} \leq \left\{ \frac{2}{\epsilon} c_\delta \left(\frac{\epsilon}{2} \right)^{\delta/q} \|(\eta_{2\epsilon}a, \eta_\epsilon f)\|_{(p,p_0)}^\delta \right\}^{1/\delta} \\ & \leq c \prod_{j=1}^n \left\{ \frac{1}{8\epsilon} \int_{x-4\epsilon}^{x+4\epsilon} |a_j(t)|^{p_j} dt \right\}^{1/p_j} \left\{ \frac{1}{4\epsilon} \int_{x-2\epsilon}^{x+2\epsilon} |f(t)|^{p_0} dt \right\}^{1/p_0} \leq cA_{(p,p_0)}(a, f)(x). \end{aligned}$$

Since $\delta < \delta_1 \Rightarrow A_\delta(f)(x) \leq A_{\delta_1}(f)(x)$ we get the estimate

$$|T_*^m(a, f)(x)| \leq c[A_\delta(T^m(a, f)(x)) + A_{(p,p_0)}(a, f)(x)].$$

The weak type estimate for T_*^m then follows from (4.7) and

(4.8) OBSERVATION. If T_1, \dots, T_n are weak type (p_j, p_j) operators with $1/q = \sum_{j=1}^n (1/p_j)$ and $\infty > p_j \geq 1$ for $j = 1, 2, \dots, n$, then $\prod_{j=1}^n |T_j a_j|$ satisfies the weak estimate of (4.4).

(4.9) LEMMA. If $(a, f) \in L^{(p,p_0)}$, where $1/q = \sum_{j=0}^n (1/p_j)$, and (4.4) is satisfied for this set of p_j 's, then there exist constants $\gamma_0 > 0$ and $c > 0$ so that for $0 < \gamma < \gamma_0$,

$$(4.10) \quad |x: T_*^m(a, f)(x) > 2\lambda, A_{(p,p_0)}(a, f)(x) \leq \gamma\lambda| \leq c\gamma^q |x: T_*^m(a, f)(x) > \lambda|.$$

Proof. $\{x: T_*^m(a, f)(x) > \lambda\} = \bigcup_j I_j$ where the I_j 's are open disjoint intervals, $I_j = (a_j, a_j + \delta_j)$ and $T_*^m(a, f)(a_j) \leq \lambda$ since $a_j \notin \bigcup_j I_j$.

It will suffice to establish (4.10) for each I_j since the I_j 's are disjoint. Choose an I_j with a point z satisfying $A_{(p,p_0)}(a, f)(z) \leq \gamma\lambda$. If I_j contains no such point, (4.10) is automatically satisfied. Let $\bar{I}_j = (a_j - 2\delta_j, a_j + 2\delta_j)$,

$$f_1 = \chi_{\bar{I}_j} f \quad (\chi_{\bar{I}_j} \text{ is the characteristic function of the interval } \bar{I}_j),$$

$$f_2 = f - f_1.$$

For $x \in I_j$, $T_*^m(a, f_1)(x) = T_*^m(\chi_{\bar{I}_j} a, \chi_{\bar{I}_j} f)(x)$. By virtue of Lemma (4.3)

less than or equal to one, non-negative and everywhere vanishing outside the interval $[x-2\varepsilon, x+2\varepsilon]$. Then, for $x_1 \in [x-\varepsilon/4, x+\varepsilon/4]$, we get the identity

$$(4.5) \quad T_\varepsilon^m(a, f)(x) = \int [t_m(x, y) - t_m(x_1, y)][1 - \eta_\varepsilon(y)]f(y)dy + \\ + \int t_m(x_1, y)f(y)dy - \int t_m(x_1, y)\eta_\varepsilon(y)f(y)dy + \\ + \int t_m(x, y)[\eta_\varepsilon(y) - \chi_\varepsilon(y)]f(y)dy.$$

Taking absolute values, raising to the δ power, averaging in x_1 over the interval $[x-\varepsilon/4, x+\varepsilon/4]$ and taking the $1/\delta$ power gives the estimate

$$(4.6) \quad |T^m(a, f)(x)| \\ \leq c \left\{ (a, f)^{(*)}(x) + A_\delta(T^m(a, f))(x) + \sup_{\varepsilon > 0} \left(\frac{2}{\varepsilon} \int_{x-\varepsilon/4}^{x+\varepsilon/4} |T^m(a, \eta_\varepsilon f)(x_1)|^\delta dx_1 \right)^{1/\delta} \right\}.$$

The $(a, f)^{(*)}(x)$ comes from the first and last term of (4.5) after applying Lemma (4.2). That is,

$$\left| \int [t_m(x, y) - t_m(x_1, y)][1 - \eta_\varepsilon(y)]f(y)dy \right| \\ \leq \int |t_m(x, y) - t_m(x_1, y)|[1 - \chi_\varepsilon(y)]|f(y)|dy \leq c(a, f)^{(*)}(x)$$

and

$$\left| \int t_m(x_1, y)[\eta_\varepsilon(y) - \chi_\varepsilon(y)]f(y)dy \right| \\ \leq ca^{(*)}(x) \int_{2\varepsilon > |x-y| > \varepsilon} \frac{|x-x_1|}{|x-y|^2} |f(y)|dy \leq c(a, f)^{(*)}(x).$$

Both estimates are independent of x_1 and are left unaffected by averaging in x_1 .

To evaluate the second term on the right hand side of (4.6) we use the fact that the usual Hardy-Littlewood maximal function is weak type 1-1. If we set $E_\lambda = \{x: |T^m(a, f)(x)|^{\delta^*} > \lambda^\delta\}$, then

$$|E_\lambda| \leq \frac{2}{\lambda^\delta} \int_{E_\lambda} |T^m(a, f)(x)|^\delta dx$$

using Kolmogorov's inequality

$$\leq \frac{2}{\lambda^\delta} |E|^{1-\delta/\alpha} \|(a, f)\|_{(p, p_0)}^\delta.$$

Taking the $1/\delta$ power of both sides and simplifying we get

$$(4.7) \quad |E_\lambda| \leq \left\{ \frac{2 \|(a, f)\|_{(p, p_0)}^\alpha}{\lambda} \right\}.$$

we have for $1/q = \sum_{j=0}^n (1/p_j)$,

$$\begin{aligned}
 (4.11) \quad |x \in I_j: T_*^m(\chi_{I_j} a, \chi_{I_j} f)(x) > \beta\lambda| &\leq c \left[\frac{\|(\chi_{I_j} a, \chi_{I_j} f)\|_{(p, p_0)}}{\beta\lambda} \right]^q \\
 &\leq c \left[\prod_{k=1}^n |\bar{I}_j|^{1/p_k} \left\{ \frac{1}{|\bar{I}_j|} \int_{\bar{I}_j} |a_k(t)|^{p_k} dt \right\}^{1/p_k} \right] \left[|\bar{I}_j|^{1/p_0} \left\{ \frac{1}{|\bar{I}_j|} \int_{\bar{I}_j} |f(t)|^{p_0} dt \right\}^{1/p_0} \right] \\
 &\leq c \left\{ \frac{A_{(p, p_0)}(a, f)(z) |\bar{I}_j|^{\sum_{j=0}^n (1/p_j)}}{\beta\lambda} \right\}^q \leq c(\gamma/\beta)^q |I_j|.
 \end{aligned}$$

For the f_2 part

$$\begin{aligned}
 (4.12) \quad |T_\varepsilon^m(a, f_2)(x)| &\leq |T_\varepsilon^m(a, f_2)(x) - T_\varepsilon^m(a, f_2)(a_j)| + |T_\varepsilon^m(a, f_2)(a_j)| \\
 &\leq \lambda + \left| \int [t_m(x-y) - t_m(a_j, y)] f_2(y) dy \right| + \\
 &\quad + \int_{x-\varepsilon}^{a_j-\varepsilon} |t_m(x, y)| |f_2(y)| dy + \int_{x+\varepsilon}^{a_j+\varepsilon} |t_m(x, y)| |f_2(y)| dy \\
 &\leq \lambda + c_1(a, f)^{(*)}(z) \leq \lambda + c_1 \gamma \lambda.
 \end{aligned}$$

The estimate for the first integral in (4.12) follows from Lemma (4.2). The estimate for the second and third integral follow from the observation that

$$|t_m(x, y)| \leq c \frac{a^{(*)}(z)}{|I_j| + \varepsilon}.$$

Combining the f_1 and f_2 estimates,

$$\begin{aligned}
 |x \in I_j: T^m(a, f_1)(x) > 2\lambda, A_{(p, p_0)}(a, f)(x) \leq \gamma\lambda| \\
 &\leq |x \in I_j: T_*^m(a, f_1)(x) > \lambda(1 - c_1\gamma)| + \\
 &\quad + |x \in I_j: T_*^m(a, f_2)(x) > \lambda(1 + c_1\gamma)| \\
 &\leq c(\gamma/(1 - c_1\gamma))^q |I_j|.
 \end{aligned}$$

Since the constants c_1, c are independent of the choice of I_j we can add the estimate for each interval to establish (4.10).

(4.13) COROLLARY. $\int |T_*^m(a, f)(x)|^q dx \leq c \int |A_{(p, p_0)}(a, f)(x)|^q dx$ for any $0 < q$.

(4.14) COROLLARY. For $a \in L^{(\infty)}$ (i.e. $a_j \in L^\infty, j = 1, \dots, n$) and $f \in L^{p_0}, p_0 > 1$,

$$\|T_*^m(a, f)\|_{p_0} \leq c \|a\|_{(\infty)} \|f\|_{p_0}$$

where $\|a\|_{(\infty)} = \prod_{j=1}^n \|a_j\|_{\infty}$.

Corollary (4.13) is the standard result of a good λ inequality (see

remarks at the end of this section). Corollary (4.14) follows from the fact that any function in L^∞ is locally in L^p and the fact that $A_p(f)(x) \leq \|f\|_\infty$.

To finish the proof of Proposition (4.1) we need one more lemma which extends the range of p 's.

(4.15) LEMMA. If $a_j \in L^1$ for $j = 1, 2, \dots, n$ and $f \in L^1$, then

$$|x: T^m(a, f)(x) > \lambda| \leq c \left\{ \frac{\|(a, f)\|_{(1,1)}}{\lambda} \right\}^{1/(n+1)}$$

where $\|(a, f)\|_{(1,1)} = \left\{ \prod_{j=1}^n \|a_j\|_1 \right\} \|f\|_1$.

Proof. The proof consists of showing the following. For any $j = 0, 1, \dots, n$, $\|a_1\|_1 = \|a_2\|_1 = \dots = \|a_j\|_1 = \|f\|_1 = 1$ and $\|a_{j+1}\|_\infty = \dots = \|a_n\|_\infty = 1$, then

$$(4.16) \quad |x: T^m(a, f)(x) > \lambda| \leq \frac{c}{\lambda^{1/(j+1)}}.$$

Choosing $j = n$ and using the multilinearity of $T^m(a, f)$, (4.16) will imply the lemma. We proceed by induction.

First, the estimate (4.16) is valid for $j = 0$ (i.e. for all the a_j 's in L^∞) since by Lemma (4.2)

$$|t_m(x, y) - t_m(x, y_1)| \leq ca^{(*)}(y_2) \frac{|y - y_1|}{|x - y|^2} \quad \text{for } |x - y| > 2|y - y_2|,$$

and T^m is bounded in L^2 as an operator acting on f whenever all the a_j 's are in L^∞ . By a standard Calderón-Zygmund argument as given in [9] we get that T^m is weak type 1-1 as an operator on f .

Next assume (4.16) is valid for $1, 2, \dots, j-1$ and assume $\|a_1\|_1 = \dots = \|a_j\|_1 = \|f\|_1 = 1$ and $\|a_{j+1}\|_\infty = \dots = \|a_n\|_\infty = 1$. Let $\Omega = \{x: a_j^*(x) > \lambda^{1/(j+1)}\}$. It follows that $\Omega = \bigcup_k I_k$ and $a_j = g + b$ where

- (i) The I_k 's are open and disjoint.
- (ii) $\sum_k |I_k| < c/\lambda^{1/(j+1)}$.
- (iii) $\frac{1}{|I_k|} \int_{I_k} |a_j(t)| dt \leq \lambda^{1/(j+1)}$.
- (iv) $|g(x)| \leq \lambda^{1/(j+1)}$.
- (v) $b = \sum_k b_k$ where b_k is supported in I_k .

This follows by defining g by

$$g(x) = \begin{cases} a_j(x), & x \notin \Omega, \\ \frac{1}{|I_k|} \int_{I_k} a_j(t) dt, & x \in I_k \end{cases}$$

and defining $b = a_j - g$. Then if χ_{I_k} denotes the characteristic function of the interval I_k , we define $b_k = b\chi_{I_k}$.

Let y_k be the center of the interval I_k and let $2I_k$ denote the interval which is twice the length of I_k and also centered at y_k .

For $x \notin 2I_k$ we observe:

$$(1) \quad \left| \int_y^x b_k(t)(x-t)^m dt \right| \leq 2|x-y|^m |I_k| \lambda^{1/(j+1)}$$

since

$$(2) \quad \frac{1}{|I_k|} \int_{I_k} |b_k(t)| dt \leq 2 \left\{ \frac{1}{|I_k|} \int_{I_k} |a_j(t)| dt \right\} \leq 2\lambda^{1/(j+1)}$$

$$\begin{aligned} & |T^m(a_1, \dots, a_{j-1}, b_k, a_{j+1}, \dots, a_n, f)(x)| \\ & \leq c \prod_{i=1}^{j-1} a_i^*(x) \prod_{i=j+1}^n \|a_i\|_\infty \frac{|I_k| \lambda^{1/(j+1)}}{|I_k|^2 + |x-y_k|^2} \int_{I_k} |f(t)| dt. \end{aligned}$$

(3) The map

$$L: f \rightarrow \sum_k \frac{|I_k|}{|I_k|^2 + |x-y_k|^2} \int_{I_k} |f(t)| dt$$

satisfies the following weak type estimate:

When $f \in L^1$,

$$(4.17) \quad |x \notin \bigcup_k 2I_k: Lf(x) > \lambda| \leq c \|f\|_1 / \lambda.$$

This follows from using Fubini's theorem to get

$$\begin{aligned} \int_{x \notin \bigcup_k 2I_k} |Lf(x)| dx & \leq \sum_k \int_{I_k} |f(y)| \left\{ |I_k| \int_{x \notin \bigcup_k 2I_k} \frac{dx}{|x-y_k|^2 + |I_k|^2} \right\} dy \\ & \leq c \int_{\Omega} |f(y)| dy \leq c \|f\|_1. \end{aligned}$$

This strong type result implies the weak estimate (4.17). Putting (1), (2) and (3) together we see that for $x \notin \bigcup_k 2I_k$,

$$|T(a_1, \dots, a_{j-1}, b, a_{j+1}, \dots, a_n, f)(x)| \leq ca_1^*(x), \dots, a_{j-1}^*(x) \lambda^{1/(j+1)} Lf(x).$$

So we have the product of j operators which are all weak type (1, 1) and the functions a_1, \dots, a_{j-1}, f all have an L^1 norm of one. This implies that

$$(4.18) \quad \begin{aligned} |x \notin \bigcup_k 2I_k: T^m(a_1, \dots, a_{j-1}, b, a_{j+1}, \dots, a_n, f)(x) > \lambda| \\ \leq |x \notin \bigcup_k 2I_k: ca_1^*(x), \dots, a_{j-1}^*(x) Lf(x) > \lambda^{j/(j+1)}| \\ \leq c/\lambda^{j/(j+1)}. \end{aligned}$$

This follows from observation (4.8) about weak type estimates for the product of weak type operators.

We next use the inductive hypothesis to make a weak type estimate for g .

$$(4.19) \quad |x: T^m(a_1, \dots, a_{j-1}, g, a_{j+1}, \dots, a_n, f)(x) > \lambda| \\ = \left| x: T^m \left(a_1, \dots, a_{j-1}, \frac{g}{\|g\|_\infty}, a_{j+1}, \dots, a_n, f \right) (x) > \frac{\lambda}{\|g\|_\infty} \right| \\ \leq c[\|g\|_\infty/\lambda]^{1/j} \leq c/\lambda^{1/(j+1)}.$$

Since $\|g\|_\infty \leq \lambda^{1/(j+1)}$, we have

(5) There is

$$(4.20) \quad \left| \bigcup_k 2I_k \right| \leq 2|\Omega| \leq c/\lambda^{1/(j+1)}.$$

Putting together estimates (4.18), (4.19) and (4.20) we have shown

$$|x: T^m(a, f)(x) > \lambda| \leq c/\lambda^{1/(j+1)}.$$

If the L^1 and L^∞ norms of the functions are arbitrary, by dividing by the appropriate norms and using the multilinearity of T^m we get the estimate:

$$|x: T^m(a, f)(x) > \lambda| \leq c \left\{ \frac{\|f\|_1 \prod_{i=1}^j \|a_i\|_1 \prod_{i=j+1}^n \|a_i\|_\infty}{\lambda} \right\}^{1/(j+1)}.$$

In particular, for $(a, f) \in L^{(1,1)}$,

$$|x: T^m(a, f)(x) > \lambda| \leq c \left\{ \frac{\|(a, f)\|_{(1,1)}}{\lambda} \right\}^{1/(n+1)}.$$

Proof of Theorem III. Proposition (4.1) then follows by applying Lemmas (4.3) and (4.9) to Lemma (4.15). In other words we have the estimate:

$$|x: T_*^m(a, f)(x) > 2\lambda, (a, f)^{(*)}(x) \leq \gamma\lambda| \leq c\gamma^{n+1} |x: T_*^m(a, f)(x) > \lambda|.$$

Theorem III then follows by the following argument. If $a_j \in L^1$ for $j = 1, \dots, n$ and $f \in L^1$, then $(a, f)^{(*)}(x) \simeq c/|x|^{n+1}$ as $x \rightarrow \infty$. Hence $\int_{\mathbb{R}^n} [(a, f)^{(*)}(x)]^q dx = +\infty$ for $q \leq 1/(n+1)$. For $q > 1/(n+1)$, Lemma (4.3) guarantees the existence of I_N^e where

$$I_N^e = q \int_e^N \lambda^{q-1} |x: T_*^m(a, f)(x) > \lambda| d\lambda \\ \leq c\gamma^{n+1} \int_e^N \lambda^{q-1} |x: T_*^m(a, f)(x) > \lambda| d\lambda + c \int_e^N \lambda^{q-1} |x: (a, f)^{(*)}(x) > \lambda| d\lambda \\ \leq c\gamma^{n+1} I_N^e + c_\gamma \int_{\mathbb{R}^n} [(a, f)^{(*)}(x)]^q dx.$$

Choosing γ so that $c\gamma^{n+1} \leq \frac{1}{2}$, letting $N \rightarrow \infty$ and $\varepsilon \rightarrow 0$ completes the proof of Theorem III.

By standard arguments Theorem III implies the following important corollary.

(4.21) COROLLARY. For $(a, f) \in L^{(p, p_0)}$, $\sum_{j=0}^n (1/p_j) = 1/q < n+1$, $1 < p_0 < \infty$, $1 < p_j \leq \infty$, and $T_\varepsilon^m(a, f)$ and $T_*^m(a, f)$ the operators defined in Theorem III, the following properties are satisfied:

- (1) $\|T_*^m(a, f)\|_q \leq c\|(a, f)\|_{(p, p_0)}$.
- (2) $\lim_{\varepsilon \rightarrow 0} T_\varepsilon^m(a, f)(x)$ exists almost everywhere.

References

- [1] B. M. Baishansky and R. Coifman, *On singular integrals*, Proc. Sympos. Pure Math., vol. 10, Amer. Math. Soc., Providence, R. I. (1967), pp. 1-17.
- [2] A. P. Calderón, *Commutators of singular integrals*, Proc. Nat. Acad. Sci. U.S.A. 53 (1965), pp. 1092-1099.
- [3] — *On algebras of singular integral operators*, Proc. Sympos. Pure Math., vol. 10, Amer. Math. Soc., Providence, R. I. (1967), pp. 18-55.
- [4] R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 51 (1974), pp. 241-250.
- [5] R. Coifman and Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. 212 (1975), pp. 315-331.
- [6] —, — *Commutateurs d'intégrales singulières*, Séminaire d'Analyse Harmonique, Orsay, France 1976.
- [7] —, — *Commutateurs d'intégrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier, Grenoble 1977.
- [8] R. Coifman, R. Rochberg, and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. 103 (1976), pp. 611-635.
- [9] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton 1970.

Received January 17, 1978

(1395)